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HORIZONTAL INEQUITY ESTIMATION: THE ISSUE OF CLOSE EQUALS IDENTIFICATION

COME MISURARE L'INIQUITÀ ORIZZONTALE NELL'AMBITO DELL'IMPOSTA PERSONALE

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Abstract

This paper focuses on the estimation of horizontal inequity, remaining within the framework of the *close equals groups* approach started by Aronson, Johnson and Lambert (1994), and systematised by Urban and Lambert (2008). Within the framework of the *close equals groups* the choice of bandwidth, which determines the intervals of *close equal* income units, is fundamental. Following the existing literature on the redistributive effect decomposition, we propose a new criterion for the identification of the optimal bandwidth: in this article the identification of *close equal*s *groups* is mainly oriented to the estimation of horizontal inequity. Our criterion intends to be a contribution to empirical work that focuses on comparing the effects of different tax systems on a particular population of taxpayers. In order to test the robustness of the new criterion, different tax systems, characterised by different degrees of tax progressivity, are applied to Italian and Polish personal income tax data. Our results suggest the bandwidths, chosen according to our methodology, are more stable than those obtained by maximising potential vertical effects.

JEL Classification Numbers: H23, H24

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1. Introduction

As Kakwani and Lambert (1998, p.369) observe "*Horizontal equity and vertical equity are two of the basic commands of social equity, requiring, respectively, the equal treatment of equals and the appropriately unequal treatment of unequals*".[1] Aronson, Johnson and Lambert (1994, p.262) have already stressed that horizontal equity violations should not be confused with vertical equity violations, as the former concept concerns "*the (unequal) treatment of equals*", and the latter deals with "*utility re-ranking*", which is *"an effect among unequals*". As the authors observe, horizontal and vertical equity violations should be considered for given "*specifications of the utility/income relationship*".

In their seminal article Aronson, Johnson and Lambert (1994) start from the redistributive effect, measured by the difference between pre- and post-tax income inequalities. These inequalities are measured by Gini coefficients. The authors consider the unequal treatment of equals, i.e. the horizontal inequity (*HI*), and violations of vertical equity, i.e. the re-ranking among unequals, as two different effects reducing the potential redistributive (or vertical) effect that would be attained if no violation occurred. Under this intuition, they decompose the redistribute effect into the sum of three indexes that measure the above three specified effects: potential vertical effect, horizontal inequity, and re-ranking.

[1] For a recent contribution to the debate on the concepts of social justice and equity, starting from Amartya Sen's conception, see Villani (2012).

Their original intention was to measure *HI* by splitting the pre-tax income distribution into groups of equals, that is into groups of income units with exactly the same income; as a consequence, within these groups the Gini coefficient is zero. In the transition from pre- to post-tax income distribution they then check if the income units, which had equal pre-tax incomes, maintain equal post-tax incomes. If they do not, then *HI* appears.

Post-tax within group income inequality can be checked easily by calculating within group Gini indexes: if these indexes become different from zero, taxes generate *HI*.

Consequently, having partitioned the pre-tax income distribution into groups of exact equals, after tax re-ranking would only involve income units that have different incomes in the pre-tax distribution.

However, the real data are often characterised by the lack or sparseness of units with the same income. As Urban and Lambert (2008) observe, empirical works apply the methodology suggested by Aronson, Johnson and Lambert (1994) in the context of *close equals groups* (henceforth *CEG*) - groups with "close" pre-tax income. *CEG* are created by splitting the pre-tax income distribution into contiguous intervals with the same bandwidth. In this context, *HI* should be measured by the increase of within group inequality. As Urban and Lambert (2008) recall, van de Ven, Creedy and Lambert (2001) show that an arbitrary specification of close equals groups can lead to misleading results. As a consequence, this approach faces two contrasting needs. First, in order to allow reliable estimations of the *HI*, the bandwidth should be large enough to create groups containing sufficient observations. Second, as it is difficult to refer to *close equals* when the bandwidth is large, it should be as small as possible. It should be noted that the former requirement conflicts with the latter.

Two other considerations are relevant in this kind of analysis. When the bandwidth grows, re-ranking within groups introduces some confusion into what, according to initial intentions, should be an analysis of equals. Moreover, when bandwidths are small, post-tax within group Gini indexes can be greater than corresponding pre-tax ones; the opposite happens when the bandwidths become large.

van de Ven, Creedy and Lambert (2001) suggest that *CEG* can be optimally defined in terms of class width by choosing the bandwidth which maximises the potential (vertical) redistributive effect. Urban and Lambert (2008) reconsider the previous literature on this topic, and *"reconcile a number of approaches to measuring equity in tax systems that coexist in the literature but appear to offer slightly differing recipes to the practitioner...*" (Urban - Lambert, 2008, p. 564).

Duclos, Jalbert and Araar (2003) deal with the problem of estimating *HI* by non-parametric regressions and simulations. While remaining within the *CEG* framework started by Aronson, Johnson and Lambert (1994), our analysis, aiming directly at the estimation of *HI*, reconsiders the problem of choosing the optimal bandwidth.

As outlined above, in choosing the proper bandwidth one has to cope with two contrasting requirements: *i*) in order to yield reliable estimations of *HI*, *CEG* should contain sufficient observations; *ii*) *CEG* should capture *real close* equals.

By looking inside the index, first introduced by Urban and Lambert (2008), to measure the potential (vertical) redistributive effect, we can suggest a new criterion for the identification of the optimal bandwidth when the main target is specifically oriented to the estimation of the *HI*. Our criterion balances group consistency and heterogeneity, and measures them with respect to the potential vertical effect.

We do not want to call into question the suggestion of van de Ven, Creedy and Lambert (2001), as their analysis primarily focuses on the estimation of the potential vertical effect: our criterion can be theoretically well grounded when estimating the horizontal effect.

The structure of the paper is as follows. Section 2 introduces the basic analytical instruments to face the problem and illustrates Urban and Lambert's (2008) decomposition of the redistributive effect, which is used to estimate both the vertical and the horizontal effect. In Section 3 we face the issue of choosing the optimal bandwidth, and develop an operational criterion of new conception. Investigating the effects of different tax systems on a matching population of income units, Section 4 tests the robustness of our criterion, by using the criterion proposed by van de Ven, Creedy and Lambert (2001) as a benchmark. Different tax systems characterised by different degrees of tax progressivity, as well as by different systems of tax credits, are applied to Italian and Polish pre-tax income distribution. According to our empirical results, when bandwidths are chosen by our methodology they are more stable than those obtained by maximising potential redistributive effects. Section 5 concludes.

2. Urban and Lambert's decomposition of the redistributive effect and the *HI* measure

Let $x_1, x_2, ..., x_K$ be the pre-tax income levels associated to K income units, and y_i denote the post-tax income of unit i (i=1, 2,..., K); we assume that all incomes have been transformed into equivalent incomes by a proper equivalent scale. We indicate the pre-tax and post-tax income distribution by X and Y, respectively. Let: G_X and G_Y be the Gini coefficients for the pre- and the post-tax income parade, and let D_Y be the concentration coefficient for the post-tax incomes, once they are ordered according to the corresponding pre-tax incomes, ranked in a non-decreasing order.

Urban and Lambert (2008) consider the decomposition of the redistributive effect RE:

$$RE = G_X - G_Y \,, \tag{1}$$

and the Kakwani (1984), vertical effect V^K:

$$V^K = G_X - D_Y \,. \tag{2}$$

By introducing the Atkinson-Plotnick-Kakwani index:

$$R^{APK} = G_Y - D_Y \tag{3}$$

which measures the re-ranking between the pre- and post-tax distribution,[2] *RE* can be rewritten as:

$$RE = V^K - R^{APK} \,. \tag{4}$$

Urban and Lambert split the index V^K into two indexes, measuring the "*full vertical effect*" and horizontal inequity (*HI)*, respectively. In order to split V^K, the authors adopt the approach of the close equals group (*CEG*), which are created, as said before, by partitioning the pre-tax income distribution into contiguous income classes with the same bandwidth.

Before introducing Urban and Lambert's indexes, we need to recall some properties and rules concerning the decomposition of Gini and concentration indexes, and some preliminary definitions.

As *CEG* are contiguous income groups, when considering pre-tax ordering, no overlapping exists among groups by construction, as the maximum income in group *k* (*k*=1, 2,..,*K*) cannot be greater than the minimum income in group *k*+1. As a consequence, due to the absence of any intersection among groups in the pre-tax distribution, the Gini coefficient decomposes

[2] Plotnick (1981). $0 \le R^{APK} \le 2G_Y$: it is zero when no re-ranking occurs.

exactly into the sum of G_X^B, the between group component, and G_X^W, the within group component:

$$G_X = G_X^B + G_X^W . \tag{5}$$

In (5) the between group component G_X^B is obtained by substituting all incomes within each group by their group average. The within group component is defined as $G_X^W = \sum_k a_{k,X} G_{k,X}$, where $G_{k,X}$ is the Gini coefficient for the *k*-th group, $a_{k,X}$ is the product of the population share, and the income share attributed to the *k*-th group.

Taxes can reshuffle the ordering either among income units or among income groups; leaving income units in the pre-tax ordering, if we replace pre-tax incomes by corresponding post-tax ones then the concentration index D_Y decomposes as:

$$D_Y = D_Y^B + D_Y^W . \tag{6}$$

In (6) D_Y^B is the between group concentration index: maintaining the pre-tax group ordering, it corresponds to G_X^B when the pre-tax group averages are substituted by the corresponding post-tax ones. D_Y^W is the within group concentration index, which is equal to $\sum_k a_{k,Y} D_{k,Y}$, being $D_{k,Y}$ the concentration index within group *k*, calculated when, within the group, post-tax incomes are aligned in the pre-tax order.

In general, the post-tax Gini index no longer decomposes into only two components. As Urban and Lambert explain and illustrate by examples, in the post-tax distribution, groups identified from the pre-tax distribution can now overlap one another. Having ranked *CEG* according to their post-tax income averages, it is no longer guaranteed that the maximum income of group k is no greater than the minimum income of group k+1; when this is the case group overlapping (or transvariation) is introduced by taxes. When groups overlap, the Gini coefficient decomposes as:

$$G_Y = G_Y^B + G_Y^W + R^{AJL} . \tag{7}$$

In (7) G_Y^B and G_Y^W are defined analogously to the terms G_X^B and G_X^W, which appear in (4), mutatis mutandis. In Urban and Lambert's (2008) notation, R^{AJL} is the index that measures group overlapping: R^{AJL} is zero if groups do not present any intersection, otherwise it is positive.[3]

Keeping in mind (7) and (6), the Atkinson-Plotnick-Kakwani index can be written as:

$$R^{APK} = R^{AJL} + R^B + R^W , \tag{8}$$

where R^B and R^W are given by:[4]

$$R^B = G_Y^B - D_Y^B , \tag{9}$$

$$R^W = G_Y^W - D_Y^W . \tag{10}$$

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[3] The label *AJL* is an acronym for Aronson, Johnson and Lambert (1994). Dagum (1997) calls it G^T, being T an acronym for *transvariation*.

[4] $0 \le R^B \le 2G_Y^B$: it is zero if no re-ranking occurs among group averages. $0 \le R^W \le 2G_Y^W$: it is zero if no re-ranking occurs within any group.

Urban and Lambert introduce "*a counterfactual post-tax income distribution, one in which the operation of the tax system within each close equals group has been smoothed*". In this distribution, net incomes are obtained by applying the average group tax rate to each income within a same group. Due to the properties of the Gini coefficient, the within group Gini index is calculated as:

$$G_Y^{SW} = \sum_k a_{k,Y} G_{k,X} \tag{11}$$

Observe that the concentration index for this distribution, when the pre-tax ordering is maintained, is given by the sum of the two components D_Y^B and G_Y^{SW}.

Given the definitions introduced above, Urban and Lambert measure the potential redistributive (vertical) effect by the following index:

$$V^{UL} = G_X - \left(D_Y^B + G_Y^{SW} \right). \tag{12}$$

In what concerns the horizontal effect, they suggest the index:

$$H^{UL} = D_Y^W - G_Y^{SW}. \tag{13}$$

As it is easily checked that $V^{UL} - H^{UL} = V^K$, Urban and Lambert can decompose the equation of the redistributive effect as per formula (14):

$$RE = V^{UL} - H^{UL} - R^{APK}. \tag{14}$$

Defining H^{AJL} as

$$H^{AJL} = G_Y^W - G_Y^{SW} = \sum_k a_{k,Y} \left(G_{k,Y} - G_{k,X} \right), \tag{15}$$

Urban and Lambert observe that H^{UL} can be obtained from the difference:

$$H^{UL} = H^{AJL} - R^{W} \qquad (16)$$

Whilst RE and R^{APK} are invariant, the terms V^{UL}, H^{UL}, H^{AJL} and R^{W} depend on the bandwidth, so that the choice of a proper bandwidth is crucial for a reliable estimation of these terms. Differently from Urban and Lambert, who consider the difference of the two terms H^{AJL} and R^{W}, we prefer to keep distinct the increase in within group inequality, measured by the former index, and the within group re-ranking, expressed by the latter.

H^{AJL} is a weighted sum of the differences between the post-tax and pre-tax Gini index for each group. Urban and Lambert call H^{AJL} a *pseudo-horizontal* effect, and they consider it a proper measure of *HI*, if no within group re-ranking occurred. Empirical evidence (Urban - Lambert, 2008; Mussini - Zavanella, 2009) shows that, as long as the bandwidth remains relatively narrow, H^{AJL} presents an increasing trend. In Mussini and Zavanella's empirical analysis, it starts decreasing for bandwidth larger than 5,000 euro, and in Urban and Lambert's analysis, for bandwidths larger than 15,000 HRK. In what concerns R_{Y}^{W}, it has a trend that is a direct function of the bandwidth. It is worth noting the lines that represent the behaviour of these components are not regular: they smooth when the bandwidths become relatively large.

According to the suggestion of van de Ven, Creedy and Lambert (2001), the bandwidth should be chosen in order to maximise V^{UL}. This procedure has a rationale when the main object is the estimation of the V^{UL}. However, in this paper the main issue is the estimation of the two components of H^{UL}, that is the *pseudo-horizontal* effect H^{AJL}, and R^W, which measures the extent of within group re-ranking. In the next section, by reconsidering in-depth the components of V^{UL}, we will introduce a new criterion for the choice of the bandwidth.

3. A new criterion to face the issue of the "optimal" bandwidth

Let us consider Urban and Lambert's potential (vertical) redistributive effect V^{UL} as in equation (12), and write it as:

$$V^{UL} = RE^B + R^B - P^{VW}\ , \qquad (17)$$

where $RE^B = G_X^B - G_Y^B$ and $P^{VW} = \sum_k \left(a_{k,Y} - a_{k,X}\right) \cdot G_{k,X}$.[5]

As noticed above, when estimating *HI*, as it is almost impossible to deal with a sufficient number of exact equals in order to create reasonable *CEG*, it becomes crucial to state how large the bandwidth should be. The bandwidth should be large enough to contain a sufficient number of *close equal* incomes, and small enough to avoid including rather unequal ones.

Using the approaches of Aronson, Johnson and Lambert (1994) and van de Ven, Creedy and Lambert (2001) as a reference point, let us assume that government must form and implement taxation policy subject to

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[5] P^{VW} is the acronym of *vertical within* group *progressivity*; $RE^B = G_X^B - G_Y^B$ is van de Ven, Creedy and Lambert's potential redistributive effect (V^{VCL} in Urban and Lambert's notation).

imperfection, such that after tax, incomes y_i are the sum of net incomes $v(x_i)$, deriving from the application of the effective tax schedule, and the outcome u_i of a random variable U, with zero expected value. It is the random variable that generates departures of the actual tax schedule from the effective one. If the tax schedule respects Kakwani and Lambert's (1998) axioms for the assumed utility/income relationship, according to which: (*i*) taxes should increase monotonically with respect to people's ability to pay; (*ii*) richer people should pay taxes at higher rates; (*iii*) net incomes should maintain the same ranking as pre-tax incomes, *HI* and re-ranking can only be generated by the outcomes u_i.

In order to obtain a reliable estimation for *HI*, a representative number of deviations from fair taxation should be considered. We can presume that the number of deviations is representative when their average converges to their expected value, which is zero. When within groups deviation averages $\left(1/N_g\right)\sum_{i\in g} u_i$ converge to zero, the rank of $\left[\left(1/N_g\right)\sum_{i\in g} y_i\right]$'$g$ converges to the rank of $\left[\left(1/N_g\right)\sum_{i\in g} v\left(x_i\right)\right]$'$g$, that is to the rank of $\left[\left(1/N_g\right)\sum_{i\in g} x_i\right]$: on this basis we can argue that the number of observation in *CEG* is large enough, at least in the most relevant groups, when R^B tends to zero.

However, we want these deviations to represent, not a generic inequity, but *HI*. We recall from equation (17) that the potential vertical index V^{UL}

contains the components R^B and P^{VW}; the latter depending on the within *CEG* pre-tax inequalities $G_{k,X}$ (k=1, 2, …, K), weighed by $\left(a_{k,Y} - a_{k,X}\right)$.

If considered in terms of absolute value, the weights $\left|a_{k,Y} - a_{k,X}\right|$ are a direct function of tax progressivity. They can be rewritten as $\left|n_k^2 \mu_k \left(\bar{t} - t_k\right) / n^2 \mu \left(1 - \bar{t}\right)\right|$, where n_k are the income units in group k; μ_k are the average group k pre-tax incomes in group k, t_k are the average tax rate in group k; $\bar{t}$ is the overall tax rate; and n is the total number of income units.

As a rule, since income distributions are right skewed, P^{VW} is expected to be positive. The positive values $\left(a_{k,Y} - a_{k,X}\right)$ corresponding $\bar{t} > t_k$, are generally greater than the negative ones (corresponding to $\bar{t} < t_k$), this is because the former are associated with groups more crowded than the latter. Even if, in the left tail of the distribution, *CEG* present income averages lower than those in the right tail, the frequency effect overcomes the income effect. As long as the bandwidth does not become very large, at least for all groups that can reasonably contain close equals, empirical evidence confirms that P^{VW} actually shows a trend, which is a direct function of the bandwidth, as it is for within *CEG* pre-tax inequalities $G_{k,X}$.[6] In any case, P^{VW} is never negative. Moreover, the more progressive the tax, the steeper the slope of the P^{VW} curve in its ascending part. Conversely, empirical

[6] Mussini and Zavanella (2009); Mazurek (2012).

evidence shows that R^B is an inverse function of the bandwidth, although with moderate irregular jumps.

In order to control for *CEG* consistency, R^B should be as close to zero as possible. On the other side, insofar as P^{VW} is a measure of the interaction of pre-tax within group inequalities and tax progressivity, we can control groups that are actually *CEG*, by keeping it as small as possible. In light of the specular behaviour of P^{VW} and R^B, we can determine that the most convenient bandwidth tallies with that minimising the greater index between P^{VW} and R^B.[7] As both indexes can be close to zero, as V^{UL} also tends to zero,[8] in order to rule out indeterminate results, we suggest considering the ratio of said indexes with respect to the potential vertical effect, that is $\left(P^{VW}/V^{UL}\right)$ and $\left(R^B/V^{UL}\right)$.

In light of the above argument, we can conveniently adopt the following criterion to determine the optimal bandwidth (*OB*):

$$OB = \arg \inf_B F\{B\}, \tag{18}$$

where

$$F\{B\} = \begin{cases} \dfrac{P^{VW}}{V^{UL}} & \text{if } \dfrac{P^{VW}}{V^{UL}} > \dfrac{R^B}{V^{UL}}, \\ \dfrac{R^B}{V^{UL}} & \text{otherwise,} \end{cases}$$

[7] Of course if P^{VW} increases monotonically whilst R^B decreases monotonically as the bandwidth increases, one can simply look for the bandwidth where $P^{VW} = R^B$.

[8] Not only R^B, but also P^{VW} tends to zero only for enormously large bandwidths. However, when the latter starts decreasing, V^{UL} also is much lower than V^K and tends to zero (Mussini - Zavanella, 2009; Mazurek, 2012).

and the argument B stands for bandwidth.

Resorting to criterion (18) we can determine the bandwidth, and in turn we are able to estimate the two components of H^{UL}, that is to say H^{AJL} and R^{W}. Empirical evidence shows that the new criterion exhibits an asymmetric U shape: it is quite steep on the left descending side, and presents a long, almost stable line to the right. We found that by applying criterion (18) both the bandwidths and estimates were more stable than those, for example, obtained by maximising the index of the potential vertical effect. Thus, our criterion compares favourably with the criterion which maximises both V^{UL} and $V^{AJL} = \left(G_X - G_Y^B - G_Y^{SW}\right)$, the latter being one of the main indexes discussed by Urban and Lambert for the potential vertical effect evaluation.[9]

In concluding this section, we remark on the decomposition of the Atkinson-Kakwani-Plotnick index, which the reference literature usually splits into the between group and within group re-ranking effects, together with the overlapping components, that is to say $R^{APK} = R^{AJL} + R^{B} + R^{W}$.

On the basis of Dagum's (1997) definition of the gross between group component, we prefer to decompose R^{APK} into the two effects R^{W} and R^{AG}. As Monti, Mussini and Vernizzi (2010) show, R^{AG} measures the re-ranking between income units belonging to different groups, either in the presence

[9] We do not consider maximizing V^{VCL}; as it appears in Urban and Lambert (2005; 2008), Mussini and Zavanella (2009) and Mazurek (2012). The maximum for V^{VCL} is reached for quite large bandwidths, so that income groups can hardly be considered as *CEG*. For an analysis concerning the whole income range, we refer to Mussini and Zavanella (2009) and Mazurek (2012). We observe that $P^{VW} = V^{VCL} - V^{AJL}$ and $R^{B} = V^{UL} - V^{AJL}$.

or in the absence of group average re-ranking. Conversely, R^{AJL} cannot be always interpreted as a real re-ranking indicator. In fact, when the two units *i* and *j* belong to different groups, they maintain the same relative rank positions both in the pre-tax and post-tax distribution, whilst the relative positions of their own groups permutes, they contribute to R^{AJL} but no re-ranking occurs for what concerns their pre-tax and post-tax incomes. As the authors show, R^{AG} can be calculated directly, or by subtracting R^{W} from R^{APK}.[10]

In the next section we test and confirm our criterion by applying several different tax systems to a dataset of Italian and Polish income earners. The income distributions we chose are very different; we applied a set of diverse real or hypothesised tax systems to these distributions. We show that results of our criterion are stable, and can be a useful guideline for the choice of optimal bandwidth in empirical works.

4. Features and effectiveness of the new criterion when comparing different tax systems

As the researchers are interested in analysing the effects of a real world tax reform, they must apply the same bandwidth in order to compare indexes that are functions of the bandwidth. This is in contrast to the adoption of a

[10] The label AG is the acronym of *Across Groups*. $0 \leq R^{AG} \leq R^{APK}$, the upper bound is reached when R^{W}=0.

criterion that tailors the proper bandwidth to each tax system. In fact, if one assumes that "true" indexes are calculated in correspondence with the optimal bandwidth, indexes calculated in correspondence with other bandwidths will provide, at the best, approximations of the former. Therefore, a criterion should also take into account the approximation errors that arise when a unique bandwidth is adopted for different tax systems.

To consider different hypothetical tax systems, in this section we compare our criterion as per formula (18), with the criterion which maximises the vertical effect. Even if our main interest concerns H^{AJL} and R^{W}, that is to say the two components of H^{UL}, we shall also examine the performance of our criterion in estimating both V^{UL} and V^{AJL}.[11] In the following, we indicate the values taken by (18) by *mr*, and the maximum values of V^{UL} and of V^{AJL} by *MUL* and *MAJL*, respectively.

The comparisons were performed by simulations, based on gross income distributions, in both Italy and the municipality of Wrocław (Poland). Both the Italian and Polish datasets are considered separately. The gross income distribution for Italy was obtained through a micro-simulation model based on the 2008 Bank of Italy survey of household income and wealth (Pellegrino - Piacenza - Turati, 2011), while that for the municipality of Wrocław is based on a 2001 dataset, kindly made available by the Lower-Silesian tax office.

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[11] As $R^{AG} = R^{APK} - R^{AG}$, with R^{APK} invariant with respect to the bandwidth, the considerations reported for R^{W}, can be applied also to R^{AG}.

The choice of the two datasets is based upon the argument that they exhibit different characteristics capable of highlighting the validity of the proposed criterion. The Polish pre-tax income distribution presents indexes of greater inequality, a stronger right skewness, and a heavier right tail than those of the Italian dataset.[12]

Applied economists and statisticians may then be interested in decomposing the redistributive effect by using both taxpayers' nominal income, or equivalent households' income. Here we show results concerning only nominal incomes. We also applied the same methodology to corresponding equivalent distributions, and find that results are not affected when the adopted equivalence scale varies.

A set of different hypothetical tax systems (see appendix for details) was applied to gross incomes: ten for the Italian income distribution and sixteen for the Polish one. The tax systems we chose for both countries varied from a very progressive system (21 brackets and tax rates ranging from 3 per cent to 85 per cent in the case of Italy; a system close to the Italian one in the 1970s), to a substantially flat system (in which tax progressivity depends only on tax credits and allowances for items of expenditures). The set of tax structures applied to the Polish gross incomes was constructed by taking into account tax systems actually applied or widely discussed in Poland,

[12] The summary statistics for the Italian gross income distribution are: mean income = 19,087 euro; standard deviation = 22,148; Skewness = 10.88; Kurtosis = 267.47. The summary statistics for the Polish gross income distribution are: mean income = 7,786 PLN; standard deviation = 23,180 PLN; Skewness = 24.23; Kurtosis = 825.31.

from the simplest flat tax system, to a more progressive tax system with four income brackets. In order to implement the “iniquity”, net incomes have been “disturbed” by different types of random errors. We could then generate a set of tax systems with *RE* ranging from 0.004% to 6.339 %. It should be noted that the set of tax systems we adopted are considered to have the likely characteristics of personal income tax structures applied worldwide, either presently or historically. As our chosen criterion proved to be very stable, both along different shapes of the income distribution, and along very different tax systems, it can be very useful in empirical applications.

For each criterion, *mr*, *MUL* and *MAJL*, we obtain a set of *N* bandwidths, $\left[b_1^{mr}, b_2^{mr}, .., b_i^{mr}, .., b_N^{mr}\right]$, $\left[b_1^{MUL}, b_2^{MUL}, .., b_i^{MUL}, .., b_N^{MUL}\right]$, and $\left[b_1^{MAJL}, b_2^{MAJL}, .., b_i^{MAJL}, .., b_N^{MAJL}\right]$, each bandwidth being “optimal” for one of the *N* tax systems. Table 1 and Table 2 report the series of optimal bandwidths, and the corresponding “optimal” parameter values for H^{AJL}, R^W and V^{UL}, for each criterion and each tax system.

Looking at our simulations, we see that the bandwidth range for the *mr* criterion is much narrower than those for the *MUL* and *MAJL* criteria. Consider for example the case of Italy (Table 1). Optimal bandwidth varies from 210 of system 6 to 460 of system 10 with *mr* criterion: the range of variation is 250. On the contrary, both *MAJL* and *UL* criteria show a greater range of variation and, most importantly, strong differences between them:

the optimal bandwidth varies from 100 of system 1 (the very progressive system) to 2,060 of system 2 (the flat system) for *MUL* criterion: the range of variation is 1,960. Also for *MAJL* criterion the range is larger than that of *mr* criterion: the optimal bandwidth varies from 290 of system 5 to 1,060 of system 2.

Similar results are found when simulations on the Polish distribution are considered. Table 2 shows that optimal bandwidth varies from 100 of system 7 to 320 of system 9 for *mr* criterion: the range of variation is only 220. On the contrary, both *MUL* and *MAJL* criteria show a greater range of variation: the optimal bandwidth varies from 20 of system 1 and 3 to 3,000 of system 12, and from 120 of system 3 to 3,000 of system 12, respectively.

Focusing on H^{AJL}, let us indicate by $H^{AJL}\left(b_s^{MUL} \mid s\right)$, $H^{AJL}\left(b_s^{MAJL} \mid s\right)$ and $H^{AJL}\left(b_s^{mr} \mid s\right)$ the value assumed by H^{AJL} in correspondence with the bandwidths, which are optimal (for the tax system *s*) according to the criteria *MUL*, *MAJL* and *mr*, respectively, give as "optimal" for tax-system *s*. Mutatis mutandis, the same notation is adopted to deal with R^W and V^{UL}, in correspondence with the said bandwidths.

Table 1: Italian hypothetical tax systems – G_X = 43.165

System	*RE*	R^{APK}	*Optimal bandwidth and indexes*											
			mr criterion				*MUL* criterion				*MAJL criterion*			
			Bandwidth	H^{AJL}	R^W	V^{UL}	*Bandwidth*	H^{AJL}	R^W	V^{UL}	*Bandwidth*	H^{AJL}	R^W	V^{UL}
1	5.25849	0.01404	270	0.00410	0.00409	5.27255	100	0.00188	0.00183	5.27258	780	0.00770	0.00771	5.27253
2	0.17200	0.02033	230	0.00498	0.00507	0.19226	2060	0.01704	0.01549	0.19389	2060	0.01704	0.01549	0.19389
3	3.90218	0.01721	250	0.00429	0.00441	3.91927	780	0.00925	0.00897	3.91966	780	0.00925	0.00897	3.91966
4	2.39775	0.03318	240	0.00659	0.00662	2.43089	220	0.00623	0.00613	2.43101	410	0.00988	0.00984	2.43097
5	5.05238	0.01265	240	0.00310	0.00314	5.06499	100	0.00154	0.00151	5.06506	290	0.00356	0.00359	5.06500
6	1.11956	0.02008	210	0.00447	0.00453	1.13957	780	0.01091	0.01062	1.13995	780	0.01091	0.01062	1.13995
7	3.93367	0.02461	240	0.00555	0.00555	3.95828	410	0.00821	0.00813	3.95837	410	0.00821	0.00813	3.95837
8	4.66382	0.01587	240	0.00387	0.00389	4.67967	100	0.00192	0.00188	4.67973	410	0.00561	0.00561	4.67968
9	4.76434	0.02408	220	0.00500	0.00499	4.78844	100	0.00258	0.00252	4.78847	410	0.00775	0.00780	4.78836
10	1.06943	0.43498	460	0.05091	0.05133	1.50398	780	0.08043	0.07774	1.50706	780	0.08043	0.07774	1.50706

Note: Bandwidth are expressed in euro; indexes have been multiplied by 100.
Source: Own elaborations.

Table 2: Polish hypothetical tax systems – G_X = 55.928

System	*RE*	R^{APK}	*Optimal bandwidth and indexes*											
			mr criterion				*MUL* criterion				*MAJL criterion*			
			Bandwidth	H^{AJL}	R^{W}	V^{UL}	*Bandwidth*	H^{AJL}	R^{W}	V^{UL}	*Bandwidth*	H^{AJL}	R^{W}	V^{UL}
1	6.33944	0.00608	280	0.00162	0.00175	6.34539	20	0.00018	0.00018	6.34552	280	0.00162	0.00175	6.34539
2	2.68961	0.01204	300	0.00363	0.00367	2.70160	120	0.00173	0.00172	2.70166	360	0.00413	0.00418	2.70159
3	2.87110	0.00098	120	0.00038	0.00039	2.87207	20	0.00009	0.00009	2.87208	120	0.00038	0.00039	2.87207
4	2.94569	0.00935	280	0.00293	0.00300	2.95498	80	0.00105	0.00105	2.95505	300	0.00307	0.00315	2.95496
5	2.49566	0.01111	260	0.00311	0.00318	2.50670	180	0.00234	0.00233	2.50678	360	0.00394	0.00396	2.50675
6	1.91359	0.01005	260	0.00294	0.00302	1.92356	20	0.00031	0.00031	1.92365	320	0.00343	0.00345	1.92363
7	2.87110	0.00098	100	0.00034	0.00035	2.87207	20	0.00009	0.00009	2.87208	120	0.00038	0.00039	2.87207
8	2.47797	0.00780	220	0.00224	0.00227	2.48574	60	0.00075	0.00075	2.48577	220	0.00224	0.00227	2.48574
9	0.00411	0.01025	320	0.00378	0.00381	0.01433	740	0.00622	0.00618	0.01440	740	0.00622	0.00618	0.01440
10	0.23054	0.00957	280	0.00333	0.00333	0.24010	980	0.00675	0.00661	0.24024	980	0.00675	0.00661	0.24024
11	0.17795	0.00102	220	0.00057	0.00059	0.17896	600	0.00078	0.00076	0.17899	600	0.00078	0.00076	0.17899
12	0.27561	0.00742	260	0.00268	0.00268	0.28303	3000	0.00700	0.00666	0.28337	3000	0.00700	0.00666	0.28337
13	6.33944	0.00608	280	0.00162	0.00175	6.34539	20	0.00018	0.00018	6.34552	360	0.00190	0.00202	6.34540
14	1.91359	0.01005	280	0.00312	0.00317	1.92360	20	0.00031	0.00031	1.92365	280	0.00312	0.00317	1.92360
15	5.34034	0.00065	120	0.00018	0.00020	5.34098	20	0.00005	0.00005	5.34100	280	0.00024	0.00034	5.34090
16	4.37707	0.00418	240	0.00116	0.00122	4.38119	20	0.00015	0.00015	4.38125	340	0.00143	0.00152	4.38116

Note: Bandwidth are expressed in PLN; indexes have been multiplied by 100.

Source: Own elaborations.

We consider each of the three criteria separately, and assume that, when considering the tax system *s*, the "true" values for H^{AJL} are meant to be $H^{AJL}\left(b_s^{MUL} \mid s\right)$, $H^{AJL}\left(b_s^{MAJL} \mid s\right)$ and $H^{AJL}\left(b_s^{mr} \mid s\right)$, each evaluated at the bandwidths b_s^{MUL}, b_s^{MAJL} and b_s^{mr}, respectively, which are "optimal" for the tax system *s*. For each criterion, we also calculate the quantities $H^{AJL}\left(b_i^{MUL} \mid s\right)$, $H^{AJL}\left(b_i^{MAJL} \mid s\right)$ and $H^{AJL}\left(b_i^{mr} \mid s\right)$ in correspondence to the other $\left(N-1\right)$ bandwidths (*i*=1, 2,..., *N*, *i*≠*s*), reported in Table 1 for Italy and in Table 2 for Poland. The resulting three sets of $\left(N-1\right)$ values act as approximations of the three "true" values, in the sense specified above.

The proper efficiency of criteria *MUL*, *MAJL* and *mr* can then be evaluated by the root mean square error of the approximations with respect to the corresponding "true" value. The corresponding statistics we are interested in are therefore given by:

$$\begin{aligned}
RMS^{MUL}\left\{H^{AJL} \mid s\right\} &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[H^{AJL}\left(b_i^{MUL} \mid s\right)-H^{AJL}\left(b_s^{MUL} \mid s\right)\right]^2}, \\
RMS^{MAJL}\left\{H^{AJL} \mid s\right\} &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[H^{AJL}\left(b_i^{MAJL} \mid s\right)-H^{AJL}\left(b_s^{MAJL} \mid s\right)\right]^2}, \\
RMS^{mr}\left\{H^{AJL} \mid s\right\} &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left[H^{AJL}\left(b_i^{mr} \mid s\right)-H^{AJL}\left(b_s^{mr} \mid s\right)\right]^2}.
\end{aligned} \tag{19}$$

The quantities $RMS^{MUL}\left\{H^{AJL} \mid s\right\}$, $RMS^{MAJL}\left\{H^{AJL} \mid s\right\}$ and $RMS^{mr}\left\{H^{AJL} \mid s\right\}$ can be calculated for each tax system *s*.

The relative efficiency of the *mr* criterion, with respect to *MUL* and *MAJL*, can be conveniently defined in terms of ratios:

$$\begin{aligned}
e_{mr}^{MUL} &= \left[RMS^{MUL}\left\{H^{AJL} \mid s\right\} \big/ RMS^{mr}\left\{H^{AJL} \mid s\right\}\right], \\
e_{mr}^{MAJL} &= \left[RMS^{MAJL}\left\{H^{AJL} \mid s\right\} \big/ RMS^{mr}\left\{H^{AJL} \mid s\right\}\right].
\end{aligned} \tag{20}$$

According to expression (20), *mr* is more efficient than *MUL* whenever e_{mr}^{MUL} is greater than 1. Likewise *mr* is more efficient than *MAJL* whenever e_{mr}^{MAJL} is greater than 1. As far as R^{W} (or R^{AG}) and V^{UL} are concerned, we can operate in the same way and adopt, mutatis mutandis, the same notation.

Tables 3 and 4 provide the main outcomes of the computations about the e_{mr}^{MUL}'s and e_{mr}^{MAJL}'s, for the indexes H^{AJL}, R^{W} and V^{UL}.

As we can see from Tables 3 and 4, all the e_{mr}^{MUL}'s and e_{mr}^{MAJL}'s are greater than 1 and, in most cases, even greater than 2. This means that the approximations given by the *mr*, criterion, as per formula (18), are much closer to "true" value than those obtained by the maximisation both of V^{UL} and of V^{AJL}.

In light of the foregoing results, with the support of an empirical evidence argument, we can draw the conclusion that the criterion proposed in this paper compares favourably with the one proposed by van de Ven, Creedy and Lambert (2001). Our criterion can be considered an improvement in choosing a bandwidth that has to be robust with respect to changes in post-tax income distributions, as is the case when comparing a sequence of tax reforms concerning a population of similar taxpayers.

Table 3: Efficiency of *mr* criterion in the Italian simulated tax systems

	H^{AJL}		R^{W}		V^{UL}	
	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}
Max	10.30	7.32	9.47	6.56	14.97	14.82
min	2.55	1.76	2.47	1.73	3.33	2.45
geometric mean	5.27	2.91	5.47	3.33	9.47	8.97
n. of cases>2	10	9	10	9	10	10
n. of cases 1÷2	0	1	0	1	0	0
n. of cases 0.5÷1	0	0	0	0	0	0
n. of cases $\leq$ 0.5	0	0	0	0	0	0

Source: Own elaborations.

Table 4: Efficiency of *mr* criterion in the Polish simulated tax systems

	H^{AJL}		R^{W}		V^{UL}	
	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}	e_{mr}^{MUL}	e_{mr}^{MAJL}
Max	54.81	55.88	8.98	5.92	90.07	89.56
min	2.31	1.50	1.66	1.03	1.96	2.66
geometric mean	5.73	4.19	4.02	2.38	29.94	31.08
n. of cases>2	16	11	15	12	15	16
n. of cases 1÷2	0	5	1	4	1	0
n. of cases 0.5÷1	0	0	0	0	0	0
n. of cases $\leq$ 0.5	0	0	0	0	0	0

Source: Own elaborations.

5. Concluding Remarks

Following an approach which plunges its roots into the seminal contribution of Aronson, Johnson and Lambert (1994), in this article we consider the problem of estimating the *HI* effect, which can be introduced by a tax system. In principle, *HI* is the unequal treatment of equals. Due to the sparseness of exact equals, in the mainstream literature, systematised by Urban and Lambert (2008), exact equals are approximated by *close* equals. Groups of *close* equals are created by fractioning the pre-tax distribution into intervals with the same bandwidth. The identification of bandwidth is then crucial: van de Ven, Creedy and Lambert (2001) suggest choosing a bandwidth that maximises the potential (vertical) effect. The issue which this paper addresses is how to identify an approach that explicitly balances two conflicting requirements: group consistency and with group heterogeneity. By inspecting the components of the index which Urban and Lambert adopt to measure the potential (vertical) effect, we conclude that the two above-mentioned requirements can be balanced by observing the well-known group re-ranking effect R^B, and another component we identify as the *vertical within* effect. Beginning with these two components we propose a new criterion for the identification of the bandwidth. This is the contribution of the paper, which goes beyond the mainstream literature based on the so-called *CEG* approach.

The criterion we propose can be adopted in empirical works focused on comparing the effects of different tax systems on a population of taxpayers. According to the simulations reported in Section 4, the bandwidths based on the new criterion present lower approximation errors than bandwidths based on maximising the potential vertical effect.

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APPENDIX

Two different approaches were adopted to obtain different unequal treatments of equals and re-ranking. The ten tax structures applied to the Italian case consider rate schedules, and the actual tax allowances and tax credits for items of expenditure, as well as income-related tax credits. On the other hand, the simulations performed on the Polish dataset are based on four basic tax systems applied to real gross incomes, each disturbed by adding a random term, so that sixteen different tax structures were considered.

The tax structures hypothesised for Italy are as follows:

SYSTEM 1. A very progressive system with 21 brackets and tax rates ranging from 3 per cent to 85 per cent. Only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 2. A 20 per cent flat tax rate: only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 3. A system with three brackets and three tax rates (10, 30 and 50 per cent): only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 4. A 30 per cent tax rate. In addition to tax allowances and tax credits for items of expenditure, an income related tax credit of 1,000 euro is added. It linearly decreases with income, and becomes zero above 100,000 euro.

SYSTEM 5. A system equal to system 3 with an income related tax credit as in system 4.

SYSTEM 6. A system equal to system 2 with an income related tax credit of 500 euro. It is linearly decreasing with income, and becomes zero above 50,000 euro.

SYSTEM 7. A progressive system with 9 brackets and tax rates ranging from 10 per cent to 75 per cent. Only tax allowances and tax credits for items of expenditure are allowed.

SYSTEM 8. A system equal to system 3 with an income related tax credit as in system 6.

SYSTEM 9. A system equal to system 7 with an income related tax credit as in system 6.

SYSTEM 10. A 70 per cent tax rate. Only tax allowances and tax credits for items of expenditure are allowed.

The basic tax structures hypothesised for Poland are as follows:

BASIC SYSTEM 1. One 15 per cent tax rate is applied to all incomes. All taxpayers benefit from 556.02 PLN tax credit.

BASIC SYSTEM 2. A system with three income brackets: *i*) 19 per cent from 0 to 44,490 PLN, *ii*) 30 per cent from 44,490 to 85,528 PLN, *iii*) 40 per cent over 85,528 PLN. All taxpayers benefit from 586.85 PLN tax credit.

BASIC SYSTEM 3. A system with two income brackets: *i*) 18 per cent from 0 to 85,528 PLN, *ii*) 32 per cent over 85,528 PLN. All taxpayers benefit from 556.02 PLN tax credit.

BASIC SYSTEM 4. A system with four income brackets: *i)* 10 per cent from 0 to 20,000 PLN, *ii*) 20 per cent from 20,000 to 40,000 PLN, *iii*) 30 per cent from 40,000 to 90,000 PLN, iv) 40 per cent over 90,000 PLN. All taxpayers benefit from 500.00 PLN tax credit.

For each tax payer, the tax $T(x_i)$ that results after the application of a basic tax system is then modified by a random factor, so that net income becomes $y_i = x_i - T(x_i) + z_i \cdot T(x_i)$; the factor z_i is drawn:

(a) from the uniform distributions:

($a1$) $Z \sim U(-0.2 \div 0.2)$, ($a2$) $Z \sim U(0 \div 0.4)$;

(b) from the normal distributions:

($b1$) $Z \sim N(0\,;0.0133)$, ($b2$) $Z \sim N(0\,;0.12)$;

so that each basic system generates four sub-systems. When the normal distribution is applied, the random factor z_i is considered in absolute value; the programme did not allow incomes to become either negative or greater that $2x_i$.